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Effective-Field Theory of Spin Glasses and the Coherent-Anomaly Method. I

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A new cluster-effective-field theory of spin glasses is formulated. Basic formulas for the spin-glass transition point and the spin-glass susceptibility in the high-temperature phase are obtained. The present theory combined with the coherent-anomaly method is shown to be useful to estimate the true critical point and the nonclassical critical exponent of a spin-glass transition. Concerning the two-dimensional $\pm J$ model, we have $\gamma_s = 5.2(1)$ for $T_{SG} = 0$, which agrees well with the data by some other authors. As for the three-dimensional $\pm J$ model, the present tentative analysis gives $T_{SG} = 1.2(1)(J/k_B)$ and $\gamma_s = 4(1)$, but more extensive calculations are needed.

KEY WORDS: Effective-field theory; spin-glass transition; coherent-anomaly method; nonclassical critical exponent; $\pm J$ model.

1. INTRODUCTION

Since Edwards and Anderson[(1)] proposed a mean-field theory of spin glasses, many theoretical studies (see ref. 2 for a review) have been made to clarify its essential feature. Among others, the Sherrington–Kirkpatrick model,[(3)] a model with infinite-range interactions, is well explained by Parisi's solution[(4)] of the replica-symmetry-breaking. On the other hand, as concerns a more realistic model with short-range interactions, there remain many fundamental unsolved problems. To know the lower critical dimensionality of the spin-glass transitions and to study their nonclassical critical exponents are some of the most interesting problems[(19–23)] on spin glasses. The application of the renormalization-group technique to the spin-glass transition has a difficulty. It predicts the upper critical dimensionality[(5)] $d_u = 6$, and consequently it is rather difficult by the ε-expansion method to

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know the existence of the transitions in the two- and three-dimensional systems and to study their critical exponents. For this reason, the numerical simulations[19,21–23] have been made mainly to study the criticality of spin glasses.

Recently, a new approach to critical phenomena, the coherent-anomaly method (CAM),[6–8] has been proposed. Usually, an approximation of the mean-field type yields a singularity of the response function with a classical, or Landau-type, exponent. Its residue, however, grows larger, as the singular point of the mean-field theory approaches the true critical point by improving approximations, owing to the discrepancy between the classical exponent and the nonclassical one. The CAM is a method for obtaining nonclassical critical exponents from a "coherent anomaly," namely the way the residue grows as the approximation is improved systematically (for example, as the treated clusters are enlarged). Thus it becomes important to construct a systematic series of approximations to the spin-glass transition.

In the present paper, we construct a cluster-effective-field theory of spin glasses. In Section 2, basic formulas for the spin-glass transition point and the spin-glass susceptibility are obtained. They can be improved in numerical value systematically in accord with the enlargement of the cluster size. Indeed, we observe the improvement by applying the theory to the two- and three-dimensional $\pm J$ models. In Section 3, the CAM is briefly reviewed. Applicability of the CAM to the present theory is discussed. In addition, a way to find the convergence of a series of approximations is mentioned. In Section 4, the numerical results mentioned in Section 2 are analyzed by the CAM, and the critical exponents thus obtained for the two- and three-dimensional $\pm J$ models are discussed.

2. THE FORMULATION OF EFFECTIVE-FIELD THEORY OF SPIN GLASSES

In the present section, an effective-field approximation of the spin-glass transition is formulated. An expression for the spin-glass susceptibility in this approximation is obtained. The zero of its denominator gives the spin-glass transition point T_{SG}. The numerical results obtained for some clusters of two- and three-dimensional $\pm J$ models are listed.

2.1. Effective Hamiltonians

First, an effective Hamiltonian is defined for each sample. It contains effective fields which themselves have their probability distributions.

Consider, as an original Hamiltonian, the short-range Edwards–Anderson model

$$\mathscr{H}_{\mathrm{EA}}(\{J_{ij}\}) \equiv -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j - \mu_{\mathrm{B}} H \sum_i \sigma_i \tag{1}$$

where σ's denote Ising spins. Each interaction J_{ij} has its probability distribution $P(J_{ij})$ over samples. The free energy of the total system is defined as follows:

$$F \equiv -k_{\mathrm{B}} T[\log Z(\{J_{ij}\})]_{\mathrm{av}} \tag{2}$$

where $Z(\{J_{ij}\})$ denotes the partition function of a system of a bond configuration $\{J_{ij}\}$:

$$Z(\{J_{ij}\}) \equiv \mathrm{Tr}\, e^{-\beta \mathscr{H}_{\mathrm{EA}}(\{J_{ij}\})} \qquad \text{with} \quad \beta \equiv \frac{1}{k_{\mathrm{B}} T} \tag{3}$$

and $[\cdots]_{\mathrm{av}}$ denotes the quenched average:

$$[\cdots]_{\mathrm{av}} \equiv \int \cdots \prod_{\langle ij \rangle} P(J_{ij})\, dJ_{ij} \tag{4}$$

For a system of a given bond configuration, an effective Hamiltonian of the relevant cluster Ω (Fig. 1) is defined by

$$e^{-\beta \mathscr{H}_{\mathrm{eff}}(\{J_{ij}\})} \equiv \mathrm{const} \cdot \mathrm{Tr}_{\bar{\Omega}}\, e^{-\beta \mathscr{H}_{\mathrm{EA}}(\{J_{ij}\})} \tag{5}$$

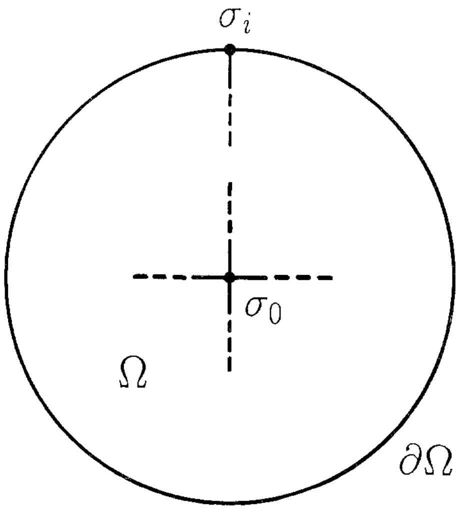

Fig. 1. The cluster Ω.

where $\mathrm{Tr}_{\bar{\Omega}}$ denotes the trace with respect to spins which do *not* belong to the cluster Ω. This effective Hamiltonian can be written generally in the form

$$\begin{aligned}\mathscr{H}_{\mathrm{eff}}=\mathscr{H}_\Omega-\mu_{\mathrm{B}}\sum_{i\in\partial\Omega}H^{(1)}_{\mathrm{eff}}(i)\,\sigma_i-\mu_{\mathrm{B}}\sum_{i,j,k\in\partial\Omega}H^{(3)}_{\mathrm{eff}}(i,j,k)\,\sigma_i\sigma_j\sigma_k-\cdots\\ -\sum_{i,j\in\partial\Omega}J^{(2)}_{\mathrm{eff}}(i,j)\,\sigma_i\sigma_j-\sum_{i,j,k,l\in\partial\Omega}J^{(4)}_{\mathrm{eff}}(i,j,k,l)\,\sigma_i\sigma_j\sigma_k\sigma_l-\cdots\end{aligned}\tag{6}$$

where

$$\mathscr{H}_\Omega\equiv-\sum_{\langle IJ\rangle\in\Omega}J_{IJ}\sigma_I\sigma_J-\mu_{\mathrm{B}}H\sum_{I\in\Omega}\sigma_I\tag{7}$$

denotes the original Hamiltonian of the cluster Ω. Each of the effective fields H_{eff} and J_{eff} has its probability distribution due to the distributions of bonds outside the cluster. The following two lemmas can be proved.

Lemma 1. Let the bond distribution $P(J_{ij})$ be symmetric. In the paramagnetic phase with no magnetic fields, the quenched averages of all the odd effective fields vanish:

$$[H^{(n)}_{\mathrm{eff}}]_{\mathrm{av}}=0\qquad\text{for}\quad n=1,3,5,\ldots\tag{8}$$

owing to the "gauge symmetry."[13]

Lemma 2. Consider a spin operator S on the cluster Ω. In the paramagnetic phase, a quenched average can be decomposed as follows:

$$[\langle S\rangle_\Omega\cdot H^{(n)}_{\mathrm{eff}}]_{\mathrm{av}}=[\langle S\rangle_\Omega]_{\mathrm{av}}\,[H^{(n)}_{\mathrm{eff}}]_{\mathrm{av}}\tag{9}$$

where

$$\langle S\rangle_\Omega\equiv\frac{\mathrm{Tr}_\Omega\,Se^{-\beta\mathscr{H}_\Omega}}{\mathrm{Tr}_\Omega\,e^{-\beta\mathscr{H}_\Omega}}\tag{10}$$

and Tr_Ω denotes the trace with respect to spins which belong to the cluster Ω.

This holds because the probability distributions of the bonds inside the cluster and those outside the cluster are independent of each other.

2.2. The One-Body-Effective-Field Approximation

Since it is impossible to determine the probability distributions of all the effective fields, some approximations must be introduced to study them explicitly. In the present paper, as an approximation, we neglect the "multi-body effective fields":

$$H_{\text{eff}}^{(n)} \equiv 0 \qquad \text{for} \quad n = 3, 5, 7,... \tag{11}$$

$$J_{\text{eff}}^{(n)} \equiv 0 \qquad \text{for} \quad n = 2, 4, 6,... \tag{12}$$

and determine the probability distributions of the "one-body effective fields" $H_{\text{eff}}^{(1)}(i)$ with a self-consistency condition. In the following, $H_{\text{eff}}^{(1)}$ is abbreviated to H_{eff}.

Let the bond distribution be a symmetric function to use the gauge symmetry. The following assumption is imposed.

Assumption 3. The probability distributions of the one-body effective fields in the paramagnetic phase can be assumed to follow a nearly Gaussian distribution, i.e.,

$$[H_{\text{eff}}^4]_{\text{av}} \sim [H_{\text{eff}}^2]_{\text{av}}^2 \sim O(H^4) \tag{13}$$

It is sufficient under this assumption to obtain the second moments of the probability distributions of the effective fields for treating the paramagnetic phase. Then we make the self-consistency condition for the effective fields as follows:

$$[\langle \sigma_0 \rangle^2]_{\text{av}} = [\langle \sigma_i \rangle^2]_{\text{av}} \qquad \forall i \in \partial\Omega \tag{14}$$

where σ_0 denotes the spin at the center of the cluster Ω. In the spin-glass phase, this assumption probably does not hold[14] and we will have to determine also higher moments: See Fig. 2.

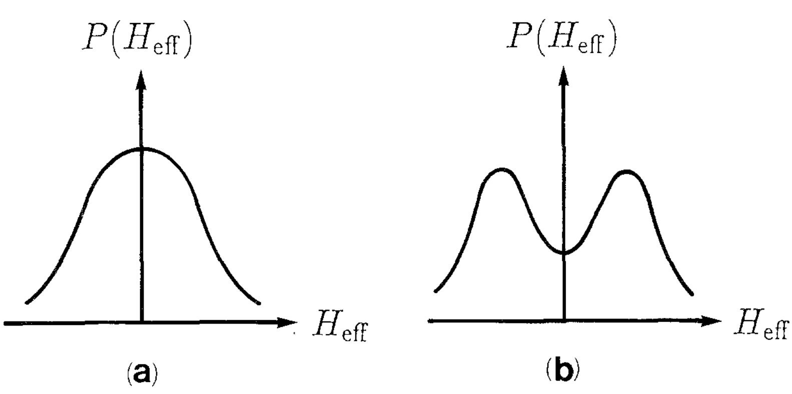

Fig. 2. Schematic forms of the probability distribution of H_{eff}: (a) $T > T_{SG}$; (b) $T < T_{SG}$.

To obtain an expression for the spin-glass susceptibility, we need to discuss Eq. (14) only[15] of the order of H^2. For a given bond configuration, we can expand $\langle\sigma_0\rangle$ and $\langle\sigma_i\rangle$ both in an applied magnetic field and effective fields as follows:

$$\langle\sigma_0\rangle = \sum_{J\in\Omega} \langle\sigma_0\sigma_J\rangle_\Omega|_{H=0} K + \sum_{j\in\partial\Omega} \langle\sigma_0\sigma_j\rangle_\Omega|_{H=0} L_j + O(K^3, K^2L, KL^2, L^3) \tag{15}$$

$$\langle\sigma_i\rangle = \sum_{J\in\Omega} \langle\sigma_i\sigma_J\rangle_\Omega|_{H=0} K + \sum_{j\in\partial\Omega} \langle\sigma_i\sigma_j\rangle_\Omega|_{H=0} L_j + O(K^3, K^2L, KL^2, L^3) \tag{16}$$

where $K \equiv \beta\mu_B H$ and $L_j \equiv \beta\mu_B H_{\text{eff}}(j)$. Here, we have set $\langle\cdots\rangle|_{H=H_{\text{eff}}=0} = \langle\cdots\rangle_\Omega|_{H=0}$, since multi-effective fields are neglected. We have also set $\langle\sigma_0\rangle_\Omega|_{H=0} = \langle\sigma_i\rangle_\Omega|_{H=0} = 0$. The remaining correlation functions can be calculated analytically. In the following, the average $\langle\cdots\rangle_\Omega|_{H=0}$ is abbreviated to $\langle\cdots\rangle_{\Omega 0}$. The expansion of $[\langle\sigma_0\rangle^2]_{\text{av}}$ gives

$$\begin{aligned}
[\langle\sigma_0\rangle^2]_{\text{av}} = & \sum_{J\in\Omega} [\langle\sigma_0\sigma_J\rangle_{\Omega 0}^2]_{\text{av}} K^2 \\
& + \sum_{I\neq J\in\Omega} [\langle\sigma_0\sigma_I\rangle_{\Omega 0} \langle\sigma_0\sigma_J\rangle_{\Omega 0}]_{\text{av}} K^2 \\
& + \sum_{j\in\partial\Omega} [\langle\sigma_0\sigma_j\rangle_{\Omega 0}^2 L_j^2]_{\text{av}} \\
& + \sum_{i\neq j\in\partial\Omega} [\langle\sigma_0\sigma_i\rangle_{\Omega 0} \langle\sigma_0\sigma_j\rangle_{\Omega 0} L_i L_j]_{\text{av}} \\
& + 2 \sum_{J\in\Omega, j\in\partial\Omega} [\langle\sigma_0\sigma_J\rangle_{\Omega 0} \langle\sigma_0\sigma_j\rangle_{\Omega 0} L_j]_{\text{av}} K \\
& + O(K^4) + O(K^2[L^2]_{\text{av}}) + O([L^4]_{\text{av}})
\end{aligned} \tag{17}$$

The quenched averages included in the third, fourth, and fifth terms can be decomposed owing to Lemma 2: (9). In addition, the second and fourth terms vanish owing to the gauge symmetry,[13] and the fifth term vanishes owing to Lemma 1: (8). The terms $O(K^2[L^2]_{\text{av}})$ and $O([L^4]_{\text{av}})$ are of the order of K^4 owing to Assumption 3: (13). The average $[\langle\sigma_i\rangle^2]_{\text{av}}$ can also be expanded similarly. Consequently, the remaining terms are

$$[\langle\sigma_0\rangle^2]_{\text{av}} = \sum_{J\in\Omega} [\langle\sigma_0\sigma_J\rangle_{\Omega 0}^2]_{\text{av}} K^2 + \sum_{j\in\partial\Omega} [\langle\sigma_0\sigma_j\rangle_{\Omega 0}^2]_{\text{av}} [L_j^2]_{\text{av}} + O(K^4) \tag{18}$$

$$[\langle\sigma_i\rangle^2]_{av} = \sum_{J\in\Omega} [\langle\sigma_i\sigma_J\rangle^2_{\Omega 0}]_{av} K^2 + \sum_{j\in\partial\Omega} [\langle\sigma_i\sigma_j\rangle^2_{\Omega 0}]_{av} [L_j^2]_{av} + O(K^4) \tag{19}$$

Substitution of (18) and (19) in the self-consistency condition (14) yields the set of equations

$$\sum_{j\in\partial\Omega} \alpha_{ij} [L_j^2]_{av} = K^2\beta_i + O(K^4) \qquad \forall i\in\partial\Omega \tag{20}$$

where

$$\begin{aligned} \alpha_{ij} &\equiv [\langle\sigma_i\sigma_j\rangle^2_{\Omega 0}]_{av} - [\langle\sigma_0\sigma_j\rangle^2_{\Omega 0}]_{av} \\ \beta_i &\equiv \sum_{J\in\Omega} \{[\langle\sigma_0\sigma_J\rangle^2_{\Omega 0}]_{av} - [\langle\sigma_i\sigma_J\rangle^2_{\Omega 0}]_{av}\} \end{aligned} \tag{21}$$

Equations (20) give the second moments of the effective fields $H_{eff}(j) = L_j/(\beta\mu_B)$ in the form

$$\begin{aligned} [L_j^2]_{av} &= K^2 \sum_{i\in\partial\Omega} (\alpha^{-1})_{ji}\, \beta_i + O(K^4) \\ &= \frac{K^2}{\det\alpha} \sum_{i\in\partial\Omega} \tilde{\alpha}_{ji}\beta_i + O(K^4) \end{aligned} \tag{22}$$

where the matrix $(\tilde{\alpha})$ denotes the cofactor matrix of the matrix (α), i.e., $(\alpha^{-1}) \equiv (\tilde{\alpha})/\det\alpha$.

As previously mentioned, it is sufficient to determine the second moments of the probability distributions of the effective fields for obtaining an expression for the spin-glass susceptibility in the paramagnetic phase. Substitution of (22) in (18) yields

$$[\langle\sigma_0\rangle^2]_{av} = \left\{ \sum_{J\in\Omega} [\langle\sigma_0\sigma_J\rangle^2_{\Omega 0}]_{av} + \frac{1}{\det\alpha} \sum_{i,j\in\partial\Omega} [\langle\sigma_0\sigma_i\rangle^2_{\Omega 0}]_{av}\, \tilde{\alpha}_{ij}\beta_j \right\} K^2 + O(K^4) \tag{23}$$

Then we arrive at the following formula for the spin-glass susceptibility:

$$\begin{aligned} \chi_{SG} &\equiv N\mu_B^2 \frac{\partial}{\partial(H^2)} [\langle\sigma_0\rangle^2]_{av} \Big|_{H=0} \\ &= N\beta^2\mu_B^4 \left\{ \sum_{J\in\Omega} [\langle\sigma_0\sigma_J\rangle^2_{\Omega 0}]_{av} + \frac{1}{\det\alpha} \sum_{i,j\in\partial\Omega} [\langle\sigma_0\sigma_i\rangle^2_{\Omega 0}]_{av}\, \tilde{\alpha}_{ij}\beta_j \right\} \end{aligned} \tag{24}$$

The second term of the above spin-glass susceptibility diverges with the critical exponent $\gamma_s = 1$. This divergence can be understood to correspond to the transition from the paramagnetic phase to the spin-glass phase. The spin-glass transition point $T_{\rm SG}$ is determined as follows:

$$\det \alpha \equiv \det |[\langle \sigma_i \sigma_j \rangle^2_{\Omega 0}]_{\rm av} - [\langle \sigma_0 \sigma_j \rangle^2_{\Omega 0}]_{\rm av}| = 0 \qquad \text{at} \quad T = T_{\rm SG} \tag{25}$$

Near and above the transition point, this spin-glass susceptibility of the type of the effective-field theory shows the following behavior:

$$\chi_{\rm SG} \simeq \bar{\chi}_{\rm SG} \frac{T_{\rm SG}}{T - T_{\rm SG}} \qquad \text{as} \quad T \to T_{\rm SG} + 0 \tag{26}$$

with

$$\bar{\chi}_{\rm SG} \equiv \frac{N\mu_{\rm B}^4}{k_{\rm B}^2 T_{\rm SG}^3} \frac{\sum_{i,j \in \partial\Omega} [\langle \sigma_0 \sigma_i \rangle^2_{\Omega 0}]_{\rm av} \tilde{\alpha}_{ij} \beta_j}{(d/dT) \det \alpha} \bigg|_{T = T_{\rm SG}} \tag{27}$$

This quantity (27) plays an important role in the CAM: See Section 3.

When all the $z_{\partial\Omega}$ sites $i \in \partial\Omega$ are located in equivalent positions in view of geometrical symmetries of the cluster Ω, we can set

$$[L^2]_{\rm av} \equiv [L_i^2]_{\rm av} \tag{28}$$

Then the expressions (24), (25), and (27) take the following rather simple forms:

$$C_0 = C_1 \qquad \text{at} \quad T = T_{\rm SG} \tag{29}$$

$$\chi_{\rm SG} = N\beta^2 \mu_{\rm B}^4 \frac{C_0 B_1 - C_1 B_0}{C_0 - C_1} \tag{30}$$

$$\bar{\chi}_{\rm SG} = \frac{N\mu_{\rm B}^4}{k_{\rm B}^2 T_{\rm SG}^3} \frac{C_0 B_1 - C_1 B_0}{d(C_0 - C_1)/dT} \bigg|_{T = T_{\rm SG}} \tag{31}$$

respectively, where

$$B_0 \equiv \sum_{J \in \Omega} [\langle \sigma_0 \sigma_J \rangle^2_{\Omega 0}]_{\rm av}, \qquad B_1 \equiv \sum_{J \in \Omega} [\langle \sigma_i \sigma_J \rangle^2_{\Omega 0}]_{\rm av} \tag{32}$$

$$C_0 \equiv \sum_{j \in \partial\Omega} [\langle \sigma_0 \sigma_j \rangle^2_{\Omega 0}]_{\rm av} = z_{\partial\Omega} [\langle \sigma_0 \sigma_j \rangle^2_{\Omega 0}]_{\rm av} \tag{33}$$

and

$$C_1 \equiv \sum_{j \in \partial\Omega} [\langle \sigma_i \sigma_j \rangle^2_{\Omega 0}]_{\rm av} = \sum_{j=1}^{z_{\partial\Omega}} [\langle \sigma_i \sigma_j \rangle^2_{\Omega 0}]_{\rm av} \tag{34}$$

for any $i \in \partial\Omega$. It can be understood[10] that Eq. (29) with (33) and (34) represents a balance between "ordering effects" and "disordering effects."

These are the results obtained in the one-body-effective-field approximation for the cluster Ω. The formulas obtained here agree with the results of Suzuki's super-effective-field theory (SEFT)[10–12] combined with the method of the "quenched real replicas."[10] For the "Bethe cluster," which is composed of a site and its nearest neighbors, these formulas are reduced to the results already obtained for the Bethe lattice.[16] The mean-field result by Edwards and Anderson,[1] which can be called the "Weiss approximation," is obtained by taking the limit $z \to \infty$[9,10] in the equations for the Bethe cluster. It is expected that the approximation is improved gradually as we calculate on larger clusters.

2.3. Numerical Results

In the following, we restrict ourselves to the $\pm J$ model, or the model with the following probability-distribution function of interactions:

$$P(J) \equiv \tfrac{1}{2}\{\delta(J - J_0) + \delta(J + J_0)\} \tag{35}$$

with $J_0 > 0$. Some of them have been already obtained by Suzuki.[10] Further calculations were done on computers by the brick-laying transfer method.[8]

The treated clusters of the two- and three-dimensional systems are listed in Fig. 3 and Fig. 4, respectively. The results obtained from these clusters are listed in Tables I and II. The data 2D-a and 3D-a are the mean-field[1] results mentioned above. It can be observed that the critical points become lower and the quantities (27) become greater as the clusters become larger.

3. THE THEORY OF THE COHERENT-ANOMALY METHOD

In the present section, the coherent-anomaly method (CAM)[6–8] proposed by Suzuki is briefly reviewed. Applicability of the CAM to the present effective-field theory is discussed. A comment concerning the convergence of a series of approximations is mentioned.

Near and above the transition point $T_{SG}^{(*)}$, the true spin-glass susceptibility is expected to show the behavior

$$\chi_{SG}^{(*)}(T) \simeq C/(T - T_{SG}^{(*)})^{\gamma_s} \qquad \text{as} \quad T \to T_{SG}^{(*)} + 0 \tag{36}$$

with the nonclassical exponent

$$\gamma_s > 1 \tag{37}$$

On the other hand, the spin-glass susceptibility of the type of (24) derived in Section 2 may yield the classical, or Landau-type, behavior

$$\chi_{\rm SG}^{(n)}(T) \simeq \bar{\chi}_{\rm SG}^{(n)} \frac{T_{\rm SG}^{(n)}}{T - T_{\rm SG}^{(n)}} \qquad \text{as} \quad T \to T_{\rm SG}^{(n)} + 0 \tag{38}$$

with

$$T_{\rm SG}^{(n)} > T_{\rm SG}^{(*)} \qquad \text{and} \qquad \gamma_s = 1 \tag{39}$$

where n specifies the size of a cluster used in an approximation.

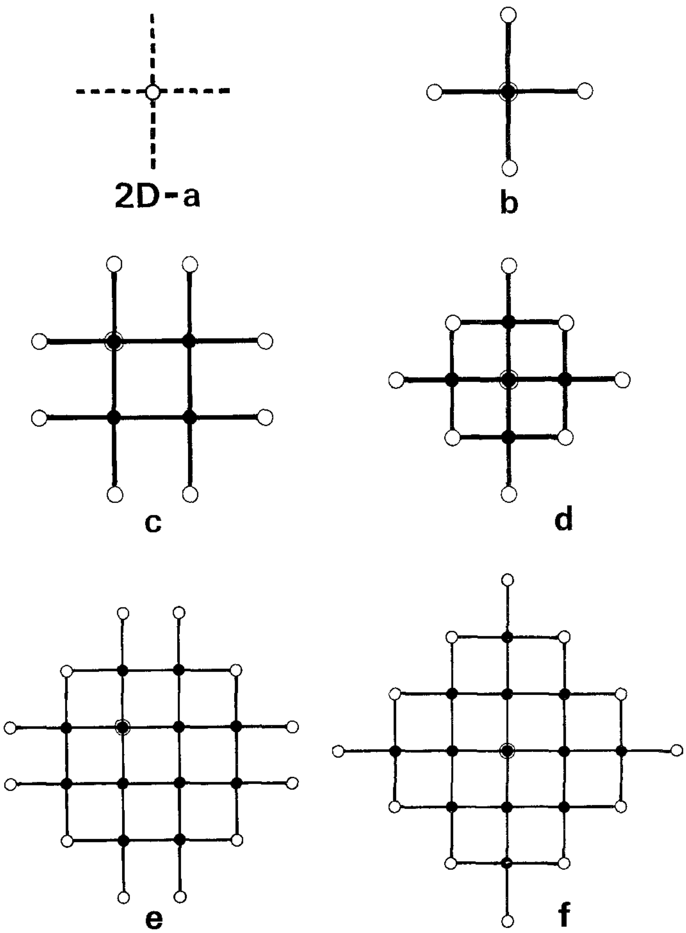

Fig. 3. Treated clusters of the two-dimensional system. The open dots denote the boundary sites of the clusters.

Consider a series of clusters $n = 1, 2, 3, \ldots$. If the series converges to the infinite system, or

$$\lim_{n \to \infty} T_{SG}^{(n)} = T_{SG}^{(*)} \tag{40}$$

$$\lim_{n \to \infty} \chi_{SG}^{(n)}(T) = \chi_{SG}^{(*)}(T) \qquad \forall T > T_{SG}^{(*)} \tag{41}$$

the discrepancy between the exponents (37) and (39) is expected to cause the anomaly of $\bar{\chi}_{SG}$:

$$\bar{\chi}_{SG}^{(n)} \to \infty \qquad \text{as} \quad n \to \infty, \qquad \text{or} \qquad T_{SG}^{(n)} \to T_{SG}^{(*)} + 0 \tag{42}$$

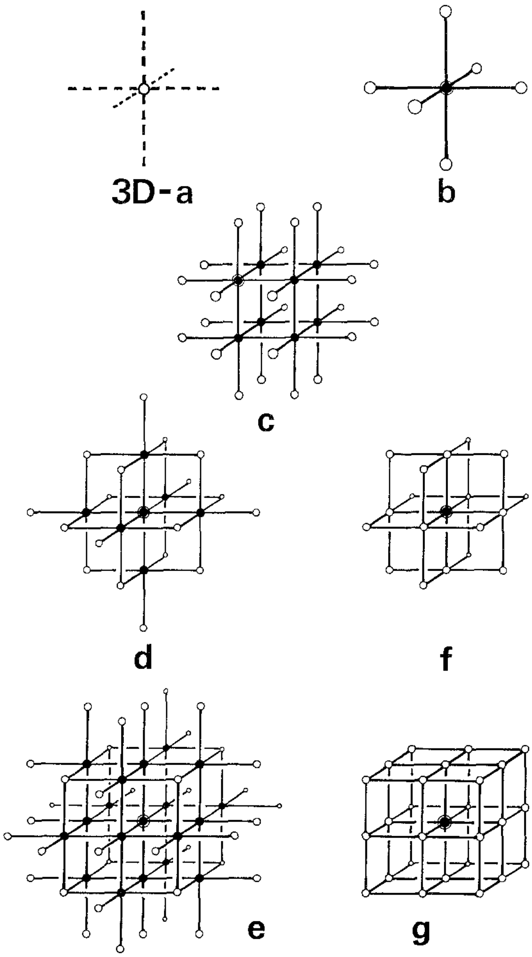

Fig. 4. Treated clusters of the three-dimensional system. The open dots denote the boundary sites of the clusters.

Table I. Data for Clusters of the Two-Dimensional $\pm J$ Model

Cluster[a]	$T_{SG}(J_0/k_B)$	$\bar{\chi}_{SG}(\mu_B^4/J_0^2)$
2D-a	2.00000	0.12500
b	1.51865	0.38017
c	1.45543	0.48918
d	1.23795	1.31524
e	1.17823	1.77188
f	1.04629	3.76646

[a] 2D-a, b,... denote the clusters in Fig. 3.

Indeed, this behavior is observed in the data listed in Tables I and II. Not only for the spin-glass, but for general phase transitions, Suzuki proposed that there may exist some systematic series of approximations for which the residue $\bar{\chi}_{SG}$ can be written in the form[6,7]

$$\bar{\chi}_{SG}^{(n)} \simeq \frac{C'}{(T_{SG}^{(n)} - T_{SG}^{(*)})^{\psi}} \quad \text{as } n \to \infty, \quad \text{or } T_{SG}^{(n)} \to T_{SG}^{(*)} + 0 \tag{43}$$

This behavior is called the coherent anomaly, and the series which shows this anomaly is called a canonical series. After some discussions[6,7] using the "envelope theory" or the "finite-degree-of-approximations scaling,"

Table II. Data for the Clusters of the Three-Dimensional $\pm J$ Model

Cluster[a]	$T_{SG}(J_0/k_B)$	$\bar{\chi}_{SG}(\mu_B^4/J_0^2)$
3D-a[9]	2.44949	0.08333
b[9]	2.07809	0.16140
c	2.03051	0.18825
d	1.92534	0.27110
e[b]	1.892(4)	0.304(3)
f	1.79015	0.50646
g[b]	1.710(5)	0.725(18)

[a] 3D-a, b,... denote the clusters in Fig. 4.

[b] For these clusters, we take the quenched averages of random samples, because of the limitation of the computational time and memory; about 0.03% out of all samples are taken for the cluster e, and about 0.17% for the cluster g.

which is analogous to Fisher's finite-size scaling theory,[17] the "coherent-anomaly relation" can be derived as follows[6,7]:

$$\gamma_s = \psi + 1 \tag{44}$$

According to the above discussions, fitting some numerical data $T_{SG}^{(1)}$, $T_{SG}^{(2)}$,... and $\bar{\chi}_{SG}^{(1)}$, $\bar{\chi}_{SG}^{(2)}$,... to the function (43) may give the true critical exponent γ_s with rather good accuracy using the relation (44). To be brief, the CAM determines the fractional critical exponents from the way in which the nonclassical properties emerge as the approximations are improved. The CAM has been applied to such critical phenomena as the ferromagnetic[6-8] and the chiral[12] transitions, and yields good results: See the references cited in ref. 11.

It is not possible at present to prove rigorously that the present series of approximations is a canonical series. The following phenomenological discussion,[7,18] however, can be made: Assume the scaling form of the spin-glass correlation as

$$[\langle \sigma_i \sigma_j \rangle^2]_{av} \sim \frac{\exp[-|i-j|/\xi(T)]}{|i-j|^{d-2+\eta}} \tag{45}$$

with

$$\xi(T) \propto (T - T_{SG}^{(*)})^{-\nu} \tag{46}$$

Now we need to distinguish between the exponent $\eta_\perp$ for correlations perpendicular to the surface of the cluster and the exponent $\eta_\parallel$ for ones parallel to it. The summations C_0 and C_1 of Eqs. (33) and (34) for a cluster of sufficiently large size L may be evaluated as follows:

$$\begin{aligned} C_0 &\equiv \sum_{j \in \partial\Omega} [\langle \sigma_0 \sigma_j \rangle_{\Omega 0}^2]_{av} \sim \int_{\partial\Omega} \frac{\exp[-R/\xi(T)]}{R^{d-2+\eta_\perp}} \, d^{d-1}R \propto L^{1-\eta_\perp} e^{-\lambda} \\ C_1 &\equiv \sum_{j \in \partial\Omega} [\langle \sigma_i \sigma_j \rangle_{\Omega 0}^2]_{av} \sim \int_{\partial\Omega} \frac{\exp[-R'/\xi(T)]}{R'^{d-2+\eta_\parallel}} \, d^{d-1}R \propto L^{1-\eta_\parallel} f_1(\lambda) \end{aligned} \tag{47}$$

where $\lambda(T) \equiv L/\xi(T)$ is a scaling variable, $R' \equiv 2L \sin(\varphi/2)$, and the function f_1 is defined by

$$f_1(\lambda) \equiv A \int_0^\pi \frac{\exp[-2\lambda \sin(\varphi/2)]}{\sin^{d-2+\eta_\parallel}(\varphi/2)} \sin\varphi \, d\varphi \tag{48}$$

with an appropriate constant A. The approximate critical point $T_{SG}(L)$ is determined by Eq. (29), or the following equation:

$$e^{\lambda_{SG}(L)} f_1(\lambda_{SG}(L)) = L^{\eta_\parallel - \eta_\perp} \tag{49}$$

where

$$\lambda_{\rm SG}(L) \equiv \frac{L}{\xi(T_{\rm SG}(L))} \tag{50}$$

The solution of this equation can be expressed as

$$T_{\rm SG}(L) - T_{\rm SG}^{(*)} \simeq L^{-1/\nu}[a + b(\eta_{\parallel} - \eta_{\perp}) \log L]^{1/\nu} \tag{51}$$

with some constants a, b, and c, as in ordinary phase transitions.[7] Then, the condition (40) may hold for the present series of approximations.

The summations of Eq. (32) can be also evaluated as the forms $B_0 \sim L^{2-\eta} f_2(\lambda)$ and $B_1 \sim L^{2-\eta} f_3(\lambda)$ with some scaling functions f_2 and f_3. The critical behavior of the anomaly (31) may be written in the following form:

$$\bar{\chi}_{\rm SG}(L) \sim \left. \frac{L^{2-\eta}[F_1(\lambda) - L^{-\Delta\eta} F_2(\lambda)]}{(1 - L^{-\Delta\eta})(d/dT)[e^{-\lambda} - f_1(\lambda)]} \right|_{\lambda = \lambda_{\rm SG}} \tag{52}$$

with $\Delta\eta \equiv |\eta_{\perp} - \eta_{\parallel}|$ and some functions F_1 and F_2. The relations

$$\lambda \equiv L(T - T_{\rm SG}^{(*)})^{\nu} \quad \text{and} \quad \frac{\partial \lambda}{\partial T} = L\nu(T - T_{\rm SG}^{(*)})^{\nu-1} \tag{53}$$

give the behavior

$$\bar{\chi}_{\rm SG}({\rm L}) \sim \left. \frac{L^{1-\eta}}{(T - T_{\rm SG}^{(*)})^{\nu-1}} \right|_{T = T_{\rm SG}(L)} \propto \frac{1}{[T_{\rm SG}(L) - T_{\rm SG}^{(*)}]^{(2-\eta)\nu-1}} \tag{54}$$

in the limit of $L \to \infty$, or the coherent anomaly (43) with $\psi = (2 - \eta)\, \nu - 1$. The coherent-anomaly relation (44) can be derived from the above behavior together with the scaling relation $\gamma_s = (2 - \eta)\, \nu$.

We make the following comment. We can know whether the "degree of approximation"[6-8]

$$\delta(T_{\rm SG}^{(n)}) \equiv \frac{T_{\rm SG}^{(n)} - T_{\rm SG}^{(*)}}{T_{\rm SG}^{(*)}} \tag{55}$$

is small enough or not in the following way: The same discussion as for (36)–(44) can be made for a temperature-dependent variable $x(T)$ instead of T, if the following expansion is possible:

$$x - x_{\rm SG} \equiv x(T) - x(T_{\rm SG}) \simeq \alpha(T - T_{\rm SG}) \tag{56}$$

If a series of approximations converges enough to show the coherent anomaly, critical data $T_{\rm SG}^{(*)}$ and ψ derived from the CAM analyses are expected not to depend on the choice of the fitting variable x. Conversely, a variety of critical data resulting from this choice can be regarded as showing that the series has not yet converged enough.

In the following analyses of the three-dimensional $\pm J$ model, the fitting variables $x = T$, $1/T$, $\tanh(1/T)$, $\tanh^2(1/T)$, and $\exp(-1/T)$ are used, where we have set $k_B = J = 1$.

4. CRITICAL DATA AND DISCUSSIONS

4.1. Results for the Two-Dimensional $\pm J$ Model

By the least-square fitting of the data for the two-dimensional $\pm J$ model (Table I), it can be concluded that $T_{SG}^{(*)} \equiv 0$.

Figure 5a suggests that we had better exclude the point 2D-a of the mean-field type. The mean-field approximation may not take part in the canonical series of the effective-field theory.

When we take both parameters ψ and $T_{SG}^{(*)}$ as free and fit the data b–f of Table I to the function

$$\bar{\chi}_{SG} = \frac{C}{(T_{SG} - T_{SG}^{(*)})^{\psi}} \tag{57}$$

we obtain

$$T_{SG}^{(*)} = 0.2 \pm 0.3 J_0/k_B \qquad \text{and} \qquad \gamma_s = 6.3 \pm 1.3 \tag{58}$$

where these errors are estimated only from the errors appearing in the least-square fitting. The critical temperature may be rather low, even if it exists.

If we assume the zero-temperature transition as in most studies[19–22] and fit the data b–f to the function (57) with $T_{SG}^{(*)} \equiv 0$, we obtain $\gamma_s = 5.25(5)$ for $T_{SG}^{(*)} \equiv 0$, where these errors are estimated as done above: See Fig. 5a. Note that in the case of the zero-temperature transition the critical exponent is obtained from the relation $\gamma_s = \tilde{\gamma}_s - 2 = \tilde{\psi} - 1$: See Appendix A. We also test the function $\log \bar{\chi}_{SG} = -\psi \log(T_{SG}) + C'$, and obtain $\gamma_s = 5.15(5)$. The naive average of them gives

$$\gamma_s = 5.2(1) \qquad \text{for} \quad T_{SG}^{(*)} \equiv 0 \tag{59}$$

This is in good agreement with the data[20] of the high-temperature expansion, $\gamma_s = 5.3(3)$ for $T_{SG}^{(*)} \equiv 0$, and with the data[21] of the Monte Carlo simulation, $\gamma_s \simeq 5.3$ for $T_{SG}^{(*)} \equiv 0$. We can also compare it with the result[22] of the numerical simulation, $\gamma_s = 4.6(5)$. Our critical data are, however, obtained from the calculations of rather small clusters. This shows the usefulness of the present theory combined with the CAM.

In connection with the above conclusion (59), we discuss here the proposition that the lower critical dimensionality is $d_l \simeq 2.5$.[24] From the phenomenology developed by McMillan,[25] it may be concluded that

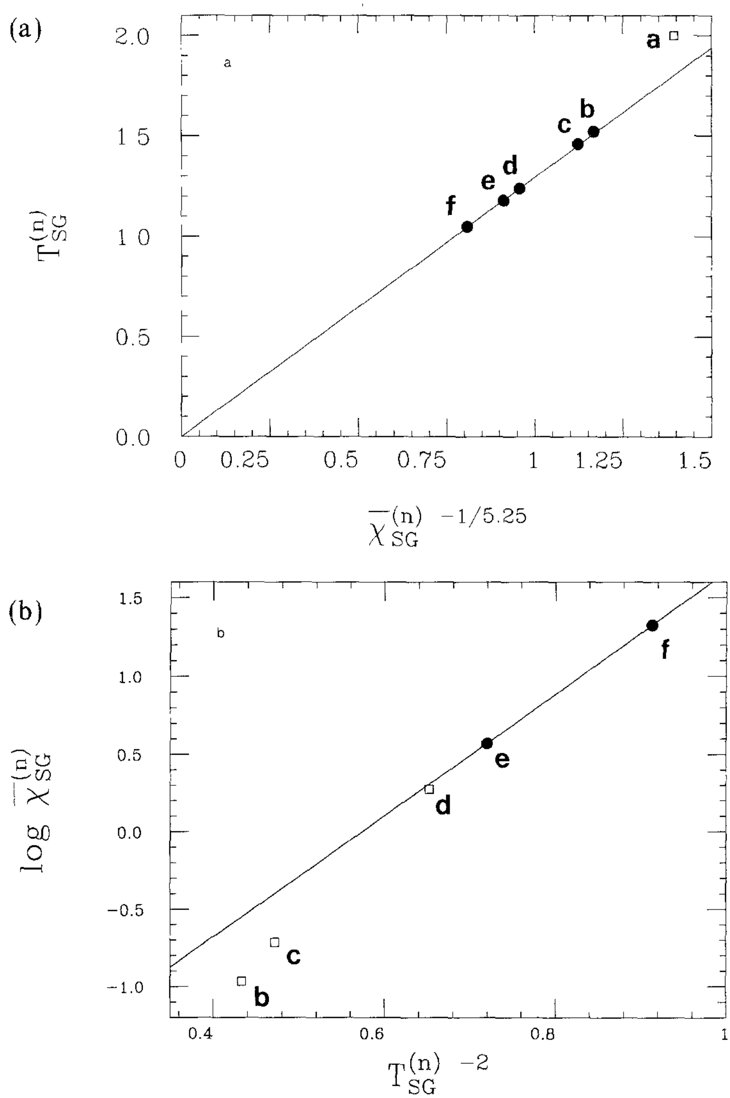

Fig. 5. The data points b–f of the two-dimensional $\pm J$ model fitted to the functions (a) $\bar{\chi}_{SG}^{(n)} = 5.0 T_{SG}^{(n)-5.25}$ and (b) $\log \bar{\chi}_{SG}^{(n)} = 3.90/(T_{SG}^{(n)})^2 - 2.24$.

the spin-glass susceptibility of systems of dimensionality d shows the following criticality:

$$\chi_{SG} \sim \begin{cases} (T - T_{SG}^{(*)})^{-\gamma_s} & \text{for} \quad d > d_l \\ e^{c/T^2} & \text{for} \quad d = d_l \\ T^{-\gamma_s} & \text{for} \quad d < d_l \end{cases} \tag{60}$$

In the case of $\chi_{SG} \sim e^{c/T^2}$, the coherent anomaly may also show the same behavior. The data $\log \bar{\chi}_{SG}^{(n)}$ vs. $1/(T_{SG}^{(n)})^2$ are plotted in Fig. 5b. The last two data points e and f yield $\log \bar{\chi}_{SG}^{(n)} = 3.90/(T_{SG}^{(n)})^2 - 2.24$. The linearity of the data is, however, worse than that of the fitting to the function (57). This observation seems to be favorable to our tentative conclusion (59) in our present approximation. Then, the present results suggest that $d_l > 2$.

4.2. Results for the Three-Dimensional $\pm J$ Model

As for the three-dimensional $\pm J$ model, although the existence of the transition point can be confirmed, it is difficult at present to conclude for it a firm value.

In least-square-fitting of these data, a difficulty arises: The data only of the clusters e and g have statistical errors because of the Monte Carlo sampling. The present analyses are made using the statistical errors for data of the clusters e and g, and using the errors appearing in the least-square-fitting for data of other clusters. This method is, however, still open to discussion, because they are different types of errors.

The fitting to the function (43) is shown in Fig. 6. The data show some irregular behavior: Not only the data points 3D-a (as in the two-dimensional system), but also the data point g does not seem to fit to the other data. As mentioned in Section 3, the data are fitted using fitting variables $x = T$, $1/T$, $\tanh(1/T)$, $\tanh^2(1/T)$, and $\exp(-1/T)$. The results obtained are shown in Table III, where the errors are estimated only from the errors appearing in the least-square-fitting. Averaging them yields

$$T_{SG}^{(*)} = 1.2(1)(J_0/k_B) \qquad \text{and} \qquad \gamma_s = 4(1) \tag{61}$$

where the errors are standard deviations. Since the data rather apart from the above average have relatively large errors, the contributions to the values (61) are small. Indeed the value of the critical point agrees with the recent results by other authors.[20,23] As for the critical exponent, however, the estimation is too rough at present to compare with them. The values of ψ obtained for some fitting variables x under the fixed values $T_{SG}^{(*)}$ are plotted in Fig. 7. The curves do not agree with each other. This suggests

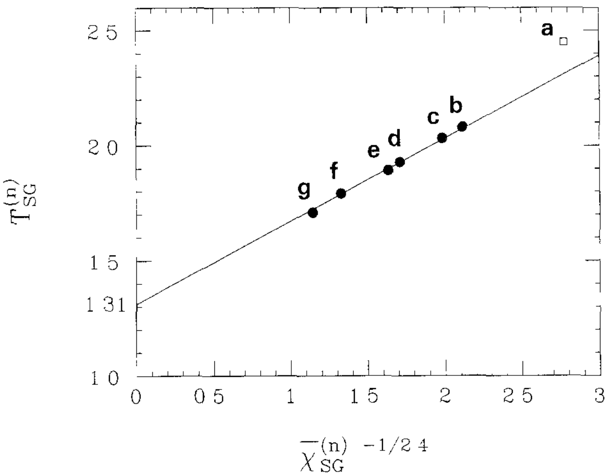

Fig. 6. The data points b–g of the three-dimensional $\pm J$ model fitted to the function $\bar{\chi}_{SG} = 0.083(T_{SG} - 1.31)^{-2.4}$.

that the series of approximations has not converged enough to show the coherent anomaly.

The fractional behavior which characterizes the three-dimensional system may emerge for the first time after more body-clusters such as the cluster g. In other words, calculations of larger clusters may be necessary to observe the coherent anomaly. Then a more efficient algorithm must be developed, because we now approach the limitation of the computational time.

Table III. The Data Points b–g for the Three-Dimensional $\pm J$ Model Fitted to the Function $\bar{\chi}_{SG}^{(n)} = C\,|x_{SG}^{(n)} - x_{SG}^{(*)}|^{-\psi}$

Fitting variable x	$T_{SG}^{(*)}$ [a]	γ_s [a]
T	1.31(6)	2.4(2)
$1/T$	1.07(13)	6.1(1.7)
$\tanh(1/T)$	1.14(12)	4.2(8)
$\tanh^2(1/T)$	0.94(22)	7.6(3.0)
$\exp(-1/T)$	1.06(13)	5.7(1.3)

[a] Errors of these date are estimated from the errors appearing in the least-square fitting to the function.

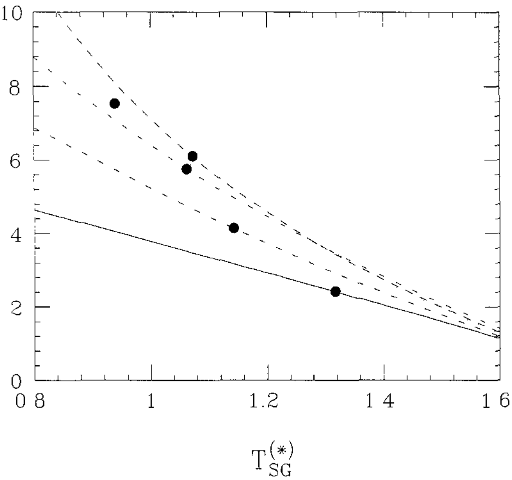

Fig. 7. The data points b–g of the three-dimensional $\pm J$ model fitted to the estimate ψ under fixed values of $T_{SG}^{(*)}$. Fitting variables are (—) T_{SG}, (- -) $1/T_{SG}$, (- - -) $\tanh(1/T_{SG})$, (···) $\tanh^2(1/T_{SG})$, and (- · · -) $\exp(-1/T_{SG})$. (●) Values of the best fittings listed in Table III.

5. CONCLUSION

An effective-field theory was constructed and the results of the mean-field and Bethe approximations were rederived. The approximations can be improved more and more, and that systematically. The calculations for some clusters and the application of the CAM to them show the usefulness of this theory. Especially for the two-dimensional $\pm J$ model, we have obtained the spin-glass transition point $T_{SG}=0$ and the critical exponent $\gamma_s = 5.2(1)$ near $T=0$, which agree with the results by other authors as well, by the calculations of rather small clusters. As for the three-dimensional $\pm J$ model, the existence of the transition can be confirmed. We may have to treat larger clusters, however (but possibly smaller than ones used in other methods), to evaluate accurate critical values by using the CAM theory. We are now planning to make Monte Carlo simulations.

APPENDIX. THE DEFINITION OF THE SPIN-GLASS SUSCEPTIBILITY

In the two-dimensional spin-glass system of Ising spins, the conclusion that $T_{SG}=0$ is now quite acceptable. There are two types of definition of the response function; the Edwards–Anderson susceptibility, as

$$\chi_{EA} \equiv \frac{1}{N} \sum_{i,j} [\langle \sigma_i \sigma_j \rangle^2]_{av} \tag{A.1}$$

and the spin-glass susceptibility, as

$$\chi_{SG} \equiv \frac{\beta^2}{N} \sum_{i,j} [\langle \sigma_i \sigma_j \rangle^2]_{av} \tag{A.2}$$

In the case of $T_{SG} = 0$, there exists a discrepancy between the critical exponents of the singularity of (A.1) and (A.2) as follows: $\chi_{EA} \propto T^{-\gamma_s}$ and $\chi_{SG} \propto T^{-\tilde{\gamma}_s}$, with

$$\tilde{\gamma}_s = \gamma_s + 2 \tag{A.3}$$

In the present paper, we use the definition χ_{SG} [see Eq. (24)], but generally the definition χ_{EA} is used.

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